Force correlations in the q model for general q distributions

Jacco H. Snoeijer and J. M. J. van Leeuwen
Instituut-Lorentz, Leiden University, P.O. Box 9506, 2300 RA Leiden, The Netherlands

(Received 14 February 2002; published 21 May 2002)

We study force correlations in the q model for granular media at infinite depth for general q distributions. We show that there are no two-point force correlations as long as q values at different sites are uncorrelated. However, higher-order correlations can persist, and if they do, they only decay with a power of the distance. Furthermore, we find the entire set of q distributions for which the force distribution factorizes. It includes distributions ranging from infinitely sharp to almost critical. Finally, we show that two-point force correlations do appear whenever there are correlations between q values at different sites in a layer; various cases are evaluated explicitly.

DOI: 10.1103/PhysRevE.65.051306 PACS number(s): 45.70.Cc, 81.05.Rm, 02.50.Ey

I. INTRODUCTION

One of the main challenges of granular media is to characterize the network of microscopic forces in a static bead pack. In order to describe the corresponding force fluctuations, Liu *et al.* [1] introduced the q model. In this model, the beads are placed on a regular lattice and the (scalar) forces are stochastically transmitted, by random fractions denoted by the symbol q. Even in its simplest version, where one assumes a uniform q distribution, it already reproduces the main feature of the experimental observations: the probability for large forces decays exponentially [1–3]. Although for this uniform q distribution the forces become totally uncorrelated, in general, correlations do persist [4]. In the present study, we investigate for which q distributions this is the case and we reveal the surprising nature of these correlations. In order to perform an analytical study, we restrict ourselves to the scalar q model and allow only correlations between q values in a layer. More sophisticated lattice models, that include the vector nature of the force and allow correlations between layers are not considered here [5].

Although the q model is particularly simple, its behavior turns out to be very rich. First of all, there is a so-called critical q distribution, that produces a force distribution that decays algebraically instead of exponentially [4,6]. It therefore forms a critical point in the space of q distributions, and its properties were recently investigated in great detail [7,8]. A second intriguing issue concerns the top-down *dynamics* of force correlations (the downward direction can be interpreted as time) [7–9]. Even if both in the initial state (top layer) and in the asymptotic state (infinite depth) all forces are uncorrelated, there will be correlations at all intermediate levels. Correlations become longer in range while their amplitudes diminish in a diffusion process, and as a result, the asymptotic force distribution is only approached algebraically [9]. This process is closely related to the subject of this study, namely, the presence of force correlations at infinite depth.

Let us recapitulate the definition of the q model. The beads are assumed to be positioned on a regular lattice. Let f_i be the force in the downward direction on the ith bead in a layer. This bead makes contact with a number of z beads in the layer below, which we indicate by the indices $i+\alpha$. The α's are displacement vectors in the lower layer as shown in Fig. 1. Bead i transmits a fraction $q_{i,\alpha}$ of the force f_i to the bead $i+\alpha$ underneath it. These fractions are taken stochastically from a distribution satisfying the constraint

$$\sum_\alpha q_{i,\alpha}=1, \tag{1}$$

which assures mechanical equilibrium in the vertical direction. So, we can write the force f_j' on the jth bead in a layer as

$$f_j'=\sum_\alpha q_{j-\alpha,\alpha} f_{j-\alpha}. \tag{2}$$

As the weights of the particles are unimportant at infinite depth, we have left out the so-called injection term. The distribution of forces at infinite depth depends on the q distribution $H(\vec{q})$, where the symbol $\vec{q}$ is a shorthand for all the $q_{i,\alpha}$ at a given layer. This $H(\vec{q})$ can be any function that is constrained by Eq. (1). If we now assume that there are no correlations between the q values at different sites, the q distribution is of the form

$$H(\vec{q})=\prod_i \eta(\vec{q}_i)\,\delta\!\left(1-\sum_\alpha q_{i,\alpha}\right), \quad \vec{q}_i=\{q_{i,\alpha}\}, \tag{3}$$

where $\eta(\vec{q}_i)$ is symmetric in its arguments $q_{i,\alpha}$. Although we will refer to these q distributions as "uncorrelated," note that there are always correlations between the $q_{i,\alpha}$ of the same site due to the δ constraint.

In the first part of this study, we show that there is only a limited set of $\eta(\vec{q}_i)$ for which the stationary force distribution can be written as a product of single-site distributions, and therefore is totally uncorrelated. This set is an extension of the set that was already identified by Coppersmith *et al.* [4]. In their extensive study, they also provided numerical evidence that, in general, correlations can persist. We will show that correlations are still absent in the second-order moments. However, higher-order correlations do exist and surprisingly enough, these turn out to decay algebraically. The results for the triangular packing and the fcc packing are

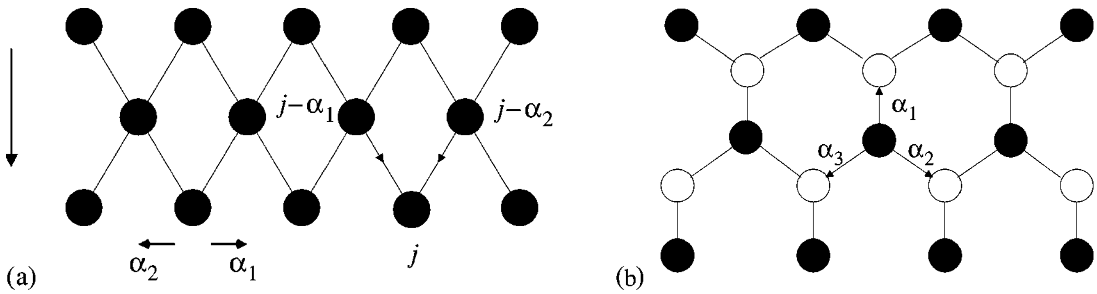

FIG. 1. The displacement vectors α in the q model for (a) the triangular packing (side view) and (b) the fcc packing (top view).

summarized in Table I, Sec. VII. In the last part of this work, we show that one induces two-point force correlations by allowing correlations between q values on different sites in a layer. These correlations will generically vanish with a power law, except for the triangular packing, where the decay of force correlations follows the decay of the q correlations.

The paper is organized as follows. In Sec. II we derive a criterion that a distribution $\eta(\vec{q}_i)$ has to obey in order to produce an uncorrelated stationary state. We then show in Sec. III, that this criterion is only obeyed for a limited set of $\eta(\vec{q}_i)$. After that, we study the nature of the correlations, by writing the evolution of the force moments as master equations in Sec. IV, and by analyzing the stationary solutions of these equations in Sec. V. Section VI deals with the effects of allowing correlations between the $\vec{q}_i$ of different sites in a layer, and the paper closes with a discussion.

II. CRITERION FOR FACTORIZATION

Using the recursive nature of the force transmission, Eq. (2), one can write down the following recursive relation for the force distribution [4,9]:

$$P'(\vec{f}') = \int H(\vec{q})d\vec{q} \int P(\vec{f})d\vec{f} \times \prod_j \delta\left(f'_j - \sum_\alpha q_{j-\alpha,\alpha} f_{j-\alpha}\right), \tag{4}$$

where we have introduced a vector notation for the forces in one layer $\vec{f} = (f_1, \ldots, f_N)$, and for the integrations we use the abbreviations

$$\int d\vec{f} = \prod_i \int_0^\infty df_i, \tag{5}$$

$$\int d\vec{q} = \prod_i \int d\vec{q}_i = \prod_i \prod_\alpha \int_0^1 dq_{i,\alpha}. \tag{6}$$

It is often convenient to work with the Laplace transform of Eq. (4). Defining the Laplace transform as

$$\tilde{P}(\vec{s}) = \int d\vec{f} \exp(-\vec{s}\cdot\vec{f}) P(\vec{f}), \tag{7}$$

the recursion simplifies to [4,9]

$$\tilde{P}'(\vec{s}) = \int H(\vec{q}) d\vec{q} \tilde{P}(\vec{s}(\vec{q})), \tag{8}$$

with

$$s_i(\vec{q}) = \sum_\alpha q_{i,\alpha} s_{i+\alpha}. \tag{9}$$

The two representations Eqs. (4) and (8) are equivalent, and they will both be used, depending on the nature of the problem.

The force distribution at infinite depth $P^*(\vec{f})$ or $\tilde{P}^*(\vec{s})$ can be obtained by finding the fixed point of the recursive relation. The main question of this section is to determine whether a given $H(\vec{q})$ leads to a $P^*(\vec{f})$ that is simply a product of single-site force distributions $p^*(f_i)$. In Sec. VI we will show that this can only be the case for q distributions of the type Eq. (3). So for this section, the question is: which $\eta(\vec{q}_i)$ led to uncorrelated asymptotic states?

To answer this question, let us assume that such a fixed point exists, i.e.,

$$P^*(\vec{f}) = \prod_i p^*(f_i), \quad \text{or} \quad \tilde{P}^*(\vec{s}) = \prod_i \tilde{p}^*(s_i). \tag{10}$$

Inserting this ansatz into the Laplace representation of the recursion relation, Eq. (8), yields

$$\tilde{P}^*(\vec{s}) = \prod_i \int \eta(\vec{q}_i)\,\delta\left(1 - \sum_\alpha q_{i,\alpha}\right) d\vec{q}_i\, \tilde{p}^*\left(\sum_\alpha q_{i,\alpha} s_{i+\alpha}\right) = \prod_i \tilde{\psi}(s_{i+\alpha_1}, \ldots, s_{i+\alpha_z}), \tag{11}$$

where the function $\tilde{\psi}(s_{i+\alpha_1}, \ldots, s_{i+\alpha_z})$ is the outcome of the integral over the $\vec{q}_i$. The arguments represent the z sites that are connected to site i in the previous layer. Integrating out all forces except those at the z sites connected to i means putting all $s_j = 0$ except the set $\{s_{i+\alpha}\}$:

$$\tilde{P}^*(s_{i+\alpha_1}, \ldots, s_{i+\alpha_z}) = \tilde{\psi}(s_{i+\alpha_1}, \ldots, s_{i+\alpha_z}) \prod_\alpha \tilde{\psi}(s_{i+\alpha}, 0, \ldots)^{z-1}. \tag{12}$$

This projection of the total force distribution can only factorize if $\tilde{\psi}(s_{i+\alpha_1},\dots,s_{i+\alpha_z})$ is a product function as well, i.e.,

$$\tilde{\psi}(s_{i+\alpha_1},\dots,s_{i+\alpha_z})=\prod_\alpha \tilde{\psi}(s_{i+\alpha}). \tag{13}$$

This leads to the following criterion for asymptotic factorization:

Given a q distribution $\eta(\vec{q})$, one can construct a factorized fixed point if, and only if, there is a function $\tilde{\psi}(s)$ that satisfies the following condition:

$$\int \eta(\vec{q})\,\delta\left(1-\sum_\alpha q_\alpha\right)d\vec{q}\left[\tilde{\psi}\left(\sum_\alpha q_\alpha s_\alpha\right)\right]^z=\prod_\alpha \tilde{\psi}(s_\alpha). \tag{14}$$

This function $\tilde{\psi}(s)$ is related to the single-site distribution as

$$\tilde{p}^*(s)=[\tilde{\psi}(s)]^z. \tag{15}$$

Here, we omitted the site index i, and furthermore, our formulation depends only on z (the number of q values per site) and not on the details of the lattice.

III. SPECIAL CLASS OF *q* DISTRIBUTIONS LEADING TO FACTORIZATION

It is a well-known fact that the so-called uniform distribution, in which $\eta(\vec{q}_i)$ is a constant, produces an uncorrelated asymptotic force distribution. In fact, Coppersmith *et al.* identified a countable set of q distributions, of which the uniform distribution is a member, that have this property [4]. Although it might seem obvious that a uniform distribution leads to an uncorrelated asymptotic state, it is really not trivial. Due to the constraint of Eq. (1), there are correlations between the $q_{i,\alpha}$ on each site i, which induce force correlations that only disappear under the special conditions discussed in the previous section, Eq. (14). In this section, we will show when these special conditions are obeyed.

There is a mathematical relation that is extremely important for the q model [10]:

$$\prod_\alpha \frac{1}{(1+s_\alpha)^r}=\frac{\Gamma(zr)}{[\Gamma(r)]^z}\int d\vec{q}\;\delta\left(1-\sum_\alpha q_\alpha\right)\times\prod_\alpha (q_\alpha)^{r-1}\frac{1}{\left(1+\sum_\alpha q_\alpha s_\alpha\right)^{zr}}. \tag{16}$$

It holds for any real $r>0$. From this relation, it is immediately clear that for all q distributions of the type

$$\eta(\vec{q})=\frac{\Gamma(zr)}{[\Gamma(r)]^z}\prod_\alpha (q_\alpha)^{r-1},\quad r>0, \tag{17}$$

there is a $\tilde{\psi}(s)$ that obeys Eq. (14), namely,

$$\tilde{\psi}(s)=\frac{1}{(1+s)^r}. \tag{18}$$

The corresponding single-site force distributions are

$$\tilde{p}^*(s)=\frac{1}{(1+s/zr)^{zr}}\quad\text{or}\quad p^*(f)=\frac{(zr)^{zr}}{\Gamma(zr)}f^{zr-1}e^{-zrf}. \tag{19}$$

We rescaled the Laplace variable s, in order to put $\langle f\rangle=1$. Coppersmith *et al.* already found these q distributions for integer values of r, also based on Eq. (16) [4]. However, it holds for any real $r>0$. This means that the set for which the stationary force distribution factorizes is substantially larger; it ranges from the infinitely sharp distribution ($r\to\infty$) to the critical distribution ($r\to 0$) [11]. Note that one recovers the results for the uniform distribution by putting $r=1$.

Although there is a huge variety of q distributions that lead to uncorrelated force distributions, in general one cannot find a $\tilde{\psi}(s)$ that obeys Eq. (14). We will prove this by making a Taylor expansion of $\tilde{\psi}(s)$,

$$\tilde{\psi}(s)=\sum_{n=0}^{\infty}\psi_n s^n, \tag{20}$$

and then try to solve for the coefficients ψ_n by imposing Eq. (14). It turns out that the equations can only be solved under special conditions, which are precisely obeyed by the class of q distributions given by Eq. (17).

Let us first focus on the left-hand side (LHS) of Eq. (14). The Taylor expansion will give rise to terms of the type $(q_1s_1)^{n_1}(q_2s_2)^{n_2}\cdots(q_zs_z)^{n_z}$, which have to be integrated over all q_α. This leads to terms $s_1^{n_1}s_2^{n_2}\cdots s_z^{n_z}$ with prefactors given by the *moments* of $\eta(\vec{q})$,

$$\overline{q_1^{n_1}q_2^{n_2}\cdots q_z^{n_z}}=\int \eta(\vec{q})\,\delta\left(1-\sum_\alpha q_\alpha\right)d\vec{q}\,q_1^{n_1}q_2^{n_2}\cdots q_z^{n_z}. \tag{21}$$

These moments are not independent, due to the constraint Eq. (1). In Appendix A, we show that the moments

$$\eta_n=\int \eta(\vec{q})\,\delta\left(1-\sum_\alpha q_\alpha\right)d\vec{q}\,q_1^n \tag{22}$$

are in fact sufficient to characterize all relevant moments of Eq. (21). Besides the moments, there are of course additional prefactors consisting of combinations of the ψ_n; these are the quantities we try to find, for a given q distribution $\eta(\vec{q})$.

The right-hand side (RHS) of Eq. (14) also produces terms $s_1^{n_1}s_2^{n_2}\cdots s_z^{n_z}$, with prefactors $\psi_{n_1}\psi_{n_2}\cdots\psi_{n_z}$. The remaining task is to equate the prefactors of the terms $s_1^{n_1}s_2^{n_2}\cdots s_z^{n_z}$ on both sides of the equation. This gives a set of equations, from which one can try to solve for the ψ_n.

The zeroth order equation is trivially obeyed for any ψ_0, as can be seen by putting all $s_\alpha=0$. For convenience we fix $\psi_0=1$. The same happens at first order, since for each α, the

LHS contains z terms $\psi_1 \overline{q_\alpha s_\alpha} = 1/z\ \psi_1 s_\alpha$, and the RHS is simply $\psi_1 s_\alpha$. The first nontrivial equation appears at second order. There are two equations, for s_α^2 and for $s_\alpha s_{\alpha'}$, where $\alpha \neq \alpha'$:

$$\left(z\psi_2 + \frac{z(z-1)}{2}\psi_1^2 \right)\eta_2 = \psi_2,$$
$$\left(z\psi_2 + \frac{z(z-1)}{2}\psi_1^2 \right)\frac{2(1-z\eta_2)}{z(z-1)} = \psi_1^2. \tag{23}$$

Due to the constraint $\Sigma_\alpha q_{i,\alpha} = 1$, one can obtain an identity by multiplying the first equation by z, and adding it to the second equation multiplied by $z(z-1)/2$. Hence, the two equations are not independent and ψ_2 can be solved. The value of ψ_2 depends only on η_2, the second moment of the q distribution [12].

Working out the combinatorics of the higher orders, one finds the following general mathematical structure:

At the nth order, there are as many equations as there are different partitions $\{n_1, n_2, \ldots, n_z\}$ that make $\Sigma_\alpha n_\alpha = n$. Permutations should not be considered as different because $\eta(\vec{q})$ is symmetric in its arguments. One of these equations is dependent, as one can obtain an identity by adding the equations, after multiplication by appropriate factors.

For $z=2$, there are two third-order equations, corresponding to the partitions $\{3,0\}$ and $\{2,1\}$, of which only one is independent. This means that ψ_3 can be solved as a function of η_2 (in Appendix A we show that η_3 depends on η_2, for $z=2$). We run into problems at fourth order, where we have $\{4,0\}$, $\{3,1\}$, and $\{2,2\}$, and hence two *a priori* independent equations for one coefficient ψ_4. It turns out that the remaining equations are only identical if there is a relation between η_4 and η_2, namely,

$$\eta_4 = \frac{30\eta_2^2 - 11\eta_2 + 1}{16\eta_2 - 2}. \tag{24}$$

In Appendix A, it is shown that this relation is precisely obeyed by the class of q distributions Eq. (17) for which $\tilde{\psi}(s)$ was already solved.

The fact that the expansion of $\tilde{\psi}(s) = [\tilde{p}^*(s)]^{1/z}$ only fails at fourth order implies that a mean field approximation, in which one explicitly assumes a product state, does give the exact results up to the third moment of $p^*(f)$. This is precisely the reason why the mean field solution $p^{\mathrm{mf}}(f)$ differs only marginally from the real solution. To be more precise, the deviation $p^{\mathrm{mf}}(f) - p^*(f)$ should change sign four times, since it does not affect all moments lower than $\langle f^4 \rangle$. A careful inspection of the numerical results in Ref. [4] for a q distribution in which $q=0.1$ or $q=0.9$ shows that these small "wiggles" are indeed present. To magnify this effect, we show our simulation data in Fig. 2.

For $z=3$, the problems already appear at third order. Since we have $\{3,0,0\}$, $\{2,1,0\}$, and $\{1,1,1\}$, we encounter two independent equations for ψ_3. Again, it turns out that the equations can be solved if there is an additional relation between the q moments:

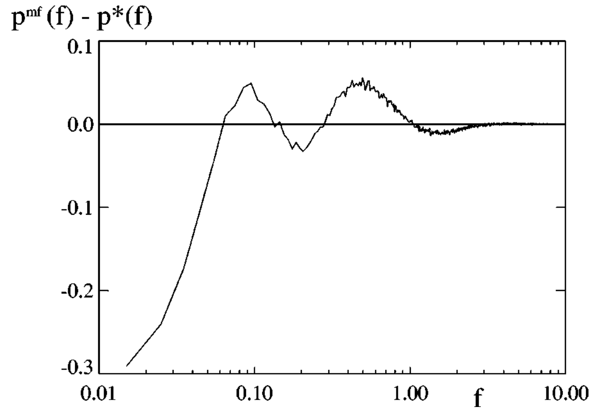

FIG. 2. Numerical simulation of a q distribution with $q=0.1$ or $q=0.9$. The small deviation $p^{\mathrm{mf}}(f) - p^*(f)$ changes sign four times.

$$\eta_3 = \frac{15\eta_2^2 - \eta_2}{9\eta_2 + 1}. \tag{25}$$

For $z>3$, there are two independent third-order equations as well, originating from $\{3,0,0,0,\ldots\}$, $\{2,1,0,0,\ldots\}$, and $\{1,1,1,0,\ldots\}$. This problem can always be overcome by assuming a particular relation between the moments η_3 and η_2, corresponding to the special q distributions of Eq. (17). Since at higher orders the number of equations per coefficient ψ_n becomes increasingly high, there will be no other q distributions than those of Eq. (17) that obey Eq. (14), and thus have an uncorrelated force distribution.

IV. EVOLUTION OF MOMENTS

Now that we know that, in general, correlations do exist in the stationary force distributions, it is interesting to study the nature of these correlations. In this section, we write the evolution of the moments as master equations, along the lines of Ref. [9]. With this formalism, we will, in the next section, analyze the correlations by finding the stationary states of these master equations.

First, let us define the second moments of a distribution as

$$M_2(k) = \langle f_i f_{i+k} \rangle = \int d\vec{f}\, f_i f_{i+k} P(\vec{f}). \tag{26}$$

We have reintroduced the site index i, and k is a displacement vector in a layer. As the system is translationally invariant, these second moments depend only on the displacement k. The recursion for these moments is obtained by combining Eqs. (2) and (4) as

$$M_2'(k) = \sum_{\alpha,\alpha'} \left(\int H(\vec{q}) d\vec{q}\, q_{j,\alpha} q_{j+k+\alpha-\alpha',\alpha'} \right) \times M_2(k+\alpha-\alpha'). \tag{27}$$

Using the overline notation for the q averages again, Eq. (27) becomes

$$M_2'(k)=\sum_{\alpha,\alpha'}\overline{q_{j,\alpha}q_{j+k+\alpha-\alpha',\alpha'}}M_2(k+\alpha-\alpha'). \quad (28)$$

This relation reveals from which points (in correlation space) the moment $M_2'(k)$ receives a contribution during a recursion step. However, it is in fact easier to consider the opposite relation, revealing how much a moment contributes to correlation space points during recursion. The "inverse" of Eq. (28) becomes

$$M_2(k)\rightarrow\overline{q_{i,\alpha}\,q_{i+k,\alpha'}}M_2'(k+\alpha'-\alpha) \quad \text{for all } \alpha,\alpha'. \quad (29)$$

This latter relation allows for a master-equation-type formulation, as we may write it in the form

$$M_2'(k)-M_2(k)=\sum_{\gamma} W_\gamma(k-\gamma)M_2(k-\gamma) - W_{-\gamma}(k)M_2(k). \quad (30)$$

The transition rates are defined as

$$W_\gamma(k)=\overline{q_{i,\alpha}q_{i+k,\alpha'}}, \quad (31)$$

with γ determined by the set α,α' as

$$\gamma=\alpha'-\alpha. \quad (32)$$

In the current problem, where we consider second-order moments, the transition rates are particularly simple. If $k\neq 0$, the q averages are independent, and will always give the value $1/z^2$ [this only holds for q distributions of the type Eq. (3)]. If $k=0$, one encounters second moments of $\eta(\vec{q})$, as in Eq. (21). This leads to the following transition rates:

$$k=0\Rightarrow W_0(0)=\eta_2, \quad W_{\gamma\neq 0}(0)=\frac{1-z\eta_2}{z(z-1)},$$

$$k\neq 0\Rightarrow W_\gamma(k)=\frac{1}{z^2}. \quad (33)$$

So, the moments evolve in an anomalous diffusion process, with differing transition rates at the origin. For a detailed discussion of the corresponding dynamics, see Ref. [9]. Note that this diffusion takes place in a $(d-1)$-dimensional space, as α, and therefore also γ, is a displacement in a layer. In the remainder of this paper we use the bold notation $\boldsymbol{\gamma}$ whenever the displacement is really a vector.

The advantage of this somewhat formal representation is that we can take it over to higher order moments without further ado. The generalization of the master equation for the nth order moments $M_n(\mathbf{r})$ becomes

$$M_n'(\mathbf{r})-M_n(\mathbf{r})=\sum_{\gamma} W_\gamma(\mathbf{r}-\boldsymbol{\gamma})M_n(\mathbf{r}-\boldsymbol{\gamma})-W_{-\gamma}(\mathbf{r})M_n(\mathbf{r}), \quad (34)$$

with the position indices $\mathbf{r}=(k_1,k_2,\ldots,k_{n-1})$, and the displacements $\boldsymbol{\gamma}$ defined as

$$\boldsymbol{\gamma}=(\alpha_1-\alpha,\alpha_2-\alpha,\ldots,\alpha_{n-1}-\alpha). \quad (35)$$

The dimensionality of the diffusion process has now become $(n-1)(d-1)$. The transition rates can be calculated as

$$W_\gamma(\mathbf{r})=\overline{q_{i,\alpha}q_{i+k_1,\alpha_1}\cdots q_{i+k_{n-1},\alpha_{n-1}}}. \quad (36)$$

Analogous to the second moments, these transition rates are all $1/z^n$, as long as the indices of the position vector $\mathbf{r}$ are not equal to zero nor coincide. However, the differing rates make the problem complicated, because one has to deal with different transition rates at special points, lines, planes, etc., in the space of diffusion.

One can now study the correlations at infinite depth by finding stationary states of the master equation for the moments. As a first attempt to construct a stationary solution, i.e., $M_n'(\mathbf{r})-M_n(\mathbf{r})=0$, one can try a detailed balance solution. Detailed balance means that there is no flow of "probability" from one point to another. In that case, all terms of the sum on the right-hand side of Eq. (34) vanish individually, i.e.,

$$W_{-\gamma}(\mathbf{r})M_n(\mathbf{r})=W_\gamma(\mathbf{r}-\boldsymbol{\gamma})M_n(\mathbf{r}-\boldsymbol{\gamma}) \quad \text{all } \mathbf{r},\boldsymbol{\gamma}. \quad (37)$$

This condition can also be formulated in terms of *elementary loops*, which are the smallest possible pathways from a point to itself. For all lattices in this study, these elementary loops are triangles, and we denote the three jump rates as (a,b,c) or (a',b',c') depending on the direction in which the loop is traversed. It is easily verified that the property

$$abc=a'b'c' \quad (38)$$

must be obeyed in *all* elementary loops in order to have a detailed balance solution. In the following section we show that correlations appear whenever the detailed balance conditions are not obeyed.

V. HIGHER-ORDER CORRELATIONS

In this section, we study the nature of the correlations for q distributions of the type Eq. (3) that do not fall into the special class of Eq. (17). We first solve the stationary master equation for the second-order moments, for which we already know that there are no correlations (Sec. III). For the triangular packing $(z=2)$, correlations only show up at fourth order, and these fall off as $1/r^5$. For $z\geqslant 3$, there are third order correlations that also decay with a power law; for the fcc packing $(z=3)$ the decay is $1/r^4$. Finally, we provide a simple relation to calculate the various exponents.

A. Second-order moments: No correlations

In order to get familiar with the structure of the master equations, we first consider the second-order moments described by Eq. (30). Away from the origin $k=0$, all transition rates of Eq. (33) are identical. Therefore, the detailed balance condition Eq. (37) requires all $M_2(k\neq 0)$ to be identical. The value at the origin $M_2(0)$ has to obey a detailed balance condition for each $\gamma \neq 0$, but these equations are identical for all γ because the corresponding rates are the same. Putting $M_2(k\neq 0)=1$, one obtains the stationary solution

$$\langle f_i f_{i+k}\rangle = \begin{cases} \dfrac{z-1}{z(1-z\eta_2)}, & k=0 \\ 1, & k\neq 0. \end{cases} \tag{39}$$

This solution precisely describes an asymptotic state without any two-point correlations, as the average of the product $\langle f_i f_j\rangle$ equals the product of the averages for all $i\neq j$. Of course, any multiple of Eq. (39), also forms a stationary solution of Eq. (30). However, these solutions are physically irrelevant in the thermodynamic limit, where the lattice size $\rightarrow\infty$ [9]. Moreover, we find that the asymptotic second force moment is solely determined by z and η_2. For critical q distributions one has $\eta_2 = 1/z$, leading to a diverging second moment.

B. Third-order moments

The diffusion of third-order moments $\langle f_i f_{i+k} f_{i+l}\rangle$ takes place on a $2(d-1)$-dimensional lattice, since there are two free parameters k and l of dimension $d-1$. On this lattice, there are three *special subspaces*, namely $k=0$, $l=0$, and $k=l$, for which the transition rates of Eq. (36) differ from the *bulk* value $1/z^3$. Moreover, the rates at the origin $k=l=0$ differ from both the bulk rates and the rates on the special subspaces.

Let us first consider the triangular packing $(z=2)$, for which the third-order moments diffuse on a two-dimensional lattice, with differing rates on three special lines. As these lines are all equivalent, it is natural to draw them at an angle of 120°, see Fig. 3. We then obtain a triangular lattice, with transitions to six nearest neighbors and two *self-jumps*, which are "transitions" to the same lattice site ($\boldsymbol{\gamma}=\mathbf{0}$). The detailed balance condition between a special line and the bulk is naturally identical to the second-order condition, implying the same ratio as in Eq. (39). As the transition rates at the origin are again identical for each $\boldsymbol{\gamma}\neq\mathbf{0}$ (because of symmetry), one can construct the following detailed balance solution:

$$\langle f_i f_{i+k} f_{i+l}\rangle = \begin{cases} \dfrac{\eta_2}{(1-2\eta_2)^2}, & \text{origin} \\ \dfrac{1}{2(1-2\eta_2)}, & \text{lines} \\ 1, & \text{bulk.} \end{cases} \tag{40}$$

This means that there are also no three-point correlations for $z=2$: at the origin we encounter $\langle f^3\rangle$, on the lines we have $\langle f_i^2 f_{i+k}\rangle = \langle f^2\rangle\langle f\rangle$, and in the bulk $\langle f_i f_{i+k} f_{i+l}\rangle = \langle f\rangle^3$. It is easily checked that condition Eq. (38) is indeed satisfied in every elementary loop.

For the fcc packing $(z=3)$, the third-order moments diffuse on a four-dimensional lattice. Unlike the $z=2$ packing,

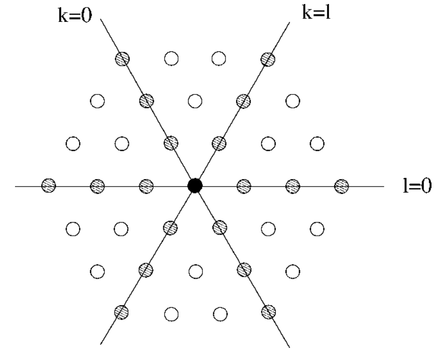

FIG. 3. Triangular packing: third-order moments diffuse on a triangular lattice.

it is not possible to construct a detailed balance solution in this case. First, we write the displacement vectors as $\boldsymbol{\gamma} = (\alpha'-\alpha, \alpha''-\alpha) = (\gamma_1, \gamma_2)$, where the α's and γ's are two-dimensional vectors (Fig. 1). One can jump away from the origin with two different rates, namely, $\overline{q_1^2 q_2}$ and $\overline{q_1 q_2 q_3}$. These rates correspond to $\gamma_1=\gamma_2$ (towards a special plane) and $\gamma_1\neq\gamma_2$ (into the bulk) respectively. Checking the detailed balance condition in the elementary triangle *origin-plane-bulk-origin*, it turns out that Eq. (38) is only obeyed if η_3 and η_2 are related as in Eq. (25). Of course, this is precisely the case for the class of Eq. (17) for which we know that asymptotic factorization occurs. In general, however, it is not possible to construct a detailed balance solution for the third-order moments. In the next paragraph, we show that the absence of detailed balance indicates that there are force correlations that decay with a power law; in this case the decay is $1/r^4$.

C. Fourth-order moments

The fourth order moments $\langle f_i f_{i+k} f_{i+l} f_{i+m}\rangle$ of the triangular packing diffuse on the bcc lattice depicted in Fig. 4. The three directions k,l,m precisely define a bcc primitive cell [13]. There are now differing rates at the origin as well as on lines and planes for which one or more indices coincide or are equal to zero. The precise geometrical structure is explained in Appendix B. There are now two *a priori* different directions away from the origin, that is to *corners* $\langle f_i^3 f_{i+1}\rangle$ and to *body centers* $\langle f_i^2 f_{i+1}^2\rangle$. Checking the loop condition Eq. (38) for the loop *origin–corner–body-center–origin*, one finds that it is only satisfied when η_4 and η_2 are related as in Eq. (24).

The question that emerges is: What are the stationary solutions of the master equation, when the detailed balance condition is frustrated at the origin? To answer this question we first consider a simplified version of the bcc problem, as a first-order approximation. In this simple version, we assume that all jump rates are $1/z^4 = 1/16$, except at the origin where we distinguish between the two different directions. Although we neglect the differing rates on the special lines and planes, the loop condition is still frustrated in the elementary loop *origin–corner–body-center–origin*. Using $\Sigma_\gamma W_\gamma(\mathbf{r})=1$, we write the stationary master equation as

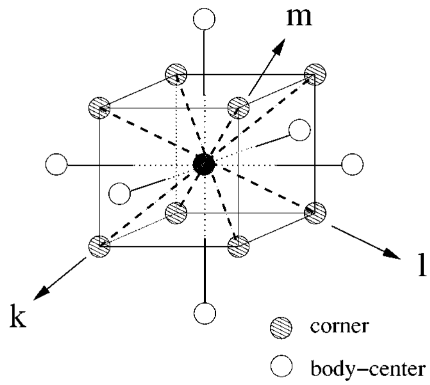

FIG. 4. Triangular packing: fourth-order moments diffuse on a bcc lattice.

$$M(\mathbf{r})=\sum_{\gamma} W_{\gamma}(\mathbf{r}-\boldsymbol{\gamma})M(\mathbf{r}-\boldsymbol{\gamma}), \tag{41}$$

or

$$[1-2W_0(\mathbf{r})]M(\mathbf{r})=\sum_{\gamma\neq 0} W_{\gamma}(\mathbf{r}-\boldsymbol{\gamma})M(\mathbf{r}-\boldsymbol{\gamma}). \tag{42}$$

This allows us to eliminate the two *self-rates* W_0 by means of a simple transformation:

$$\hat{M}(\mathbf{r})=[1-2W_0(\mathbf{r})]M(\mathbf{r}),$$

$$\hat{W}_{\gamma}(\mathbf{r})=W_{\gamma}(\mathbf{r})/[1-2W_0(\mathbf{r})]. \tag{43}$$

The sum over the new rates again adds up to unity and Eq. (42) becomes

$$\hat{M}(\mathbf{r})=\sum_{\gamma\neq 0} \hat{W}_{\gamma}(\mathbf{r}-\boldsymbol{\gamma})\hat{M}(\mathbf{r}-\boldsymbol{\gamma}). \tag{44}$$

Hence we can omit the self-jumps by first solving the equation for the "hatted" variables, and then transforming back to $M(\mathbf{r})$. As $M(\mathbf{r})\to 1$ for large r, it is convenient to write

$$\hat{M}(\mathbf{r})=\frac{7}{8}[1+\delta\hat{M}(\mathbf{r})]. \tag{45}$$

The quantity $\delta\hat{M}(\mathbf{r})$ is in fact the appropriate measure for correlations [14]. After eliminating the two self-rates, all jump rates have become 1/14, except at the origin where the rates to the eight corners (c) can differ from the rates to the six body centers (b). We, therefore, have

$$\hat{W}_{\gamma}(\mathbf{r})=1/14+\delta(\mathbf{r})\varepsilon_{\gamma}. \tag{46}$$

The rates to the corners are denoted by ε_c and those to the body centers by ε_b. They fulfill the condition $8\varepsilon_c+6\varepsilon_b=0$. This results in the following equation:

$$\delta\hat{M}(\mathbf{r})-\frac{1}{14}\sum_{\gamma\neq 0} \delta\hat{M}(\mathbf{r}-\boldsymbol{\gamma})=\frac{8}{7}\hat{M}(\mathbf{0})\sum_{\gamma\neq 0} \varepsilon_{\gamma}\delta(\mathbf{r}-\boldsymbol{\gamma}). \tag{47}$$

Note that this is a discrete version of Poisson's equation: the LHS is a discrete Laplacian and the RHS, originating from deviating rates, acts as a multipole around the origin. This equation is solved in Appendix B by a Fourier transformation, leading to

$$\hat{M}(\mathbf{r})=\frac{7}{8}+\hat{M}(0)\sum_{\mathbf{k}} \frac{E(\mathbf{k})}{1-D(\mathbf{k})}\exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{48}$$

The functions $D(\mathbf{k})$ and $E(\mathbf{k})$ are defined in Appendix B; $1-D(\mathbf{k})$ comes from the discrete Laplacian (in the continuum equation it would simply be k^2), $E(\mathbf{k})$ is the Fourier transform of the source, and the sum over $\mathbf{k}$ is the inverse Fourier transformation running over the Brillouin zone. The amplitude of the source $\hat{M}(0)$ can be obtained self-consistently, by setting $\mathbf{r}=0$. This involves a complicated integral over the Brillouin zone (BZ) of the bcc lattice; the outcome, however, will be of the order unity. The large r behavior of the correlations is determined by the small $\mathbf{k}$ behavior, so $E(\mathbf{k})/[1-D(\mathbf{k})]$ has to be expanded around $\mathbf{k}=0$. The first term that gives a contribution is

$$\frac{49\varepsilon_c}{24}\int \frac{d\mathbf{k}}{V_{\mathrm{BZ}}}\,\frac{(k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2)\exp(-i\mathbf{k}\cdot\mathbf{r})}{k^2}$$
$$\simeq\frac{343\varepsilon_c}{32\pi}\left[5\left(\frac{x^2y^2+y^2z^2+z^2x^2}{r^9}\right)-\frac{1}{r^5}\right]. \tag{49}$$

The solution of Eq. (47) decays as $1/r^5$; the terms x^2y^2, etc., give the proper angular dependence. This result can be directly understood from the analogy with electrostatics. The solution of Poisson's equation (47) can be expanded in asymptotically vanishing spherical harmonics: Y_{lm}/r^{l+1}. The symmetry of the bcc lattice allows only harmonics with $l\geqslant 4$, leading to the observed $1/r^5$ decay.

So we find that the stationary master equation for the moments becomes a discrete Poisson's equation, and the presence of differing transition rates leads to a multipole source around the location of these rates, see Eq. (47). However, this source is only "active" if there is no detailed balance, since detailed balance leads to trivial solutions like Eq. (40) [15]. Keeping this in mind, let us now investigate the real fourth-order problem, including the differing rates at the special lines and planes. We argue that the asymptotic value is still approached as $1/r^5$, but the amplitude of this field will be modified. Since there is no detailed balance, the differing rates at the lines and planes will act as sources as well. Their amplitudes, however, will decay with increasing distance, since the "flow" associated with the absence of detailed balance becomes zero at $r\to\infty$. The effect of the induced sources at the special lines and planes can be taken into account perturbatively. The first step is to only consider the effect of the origin, as we have done above. The second step would be to compute the strength of the sources at the lines and planes on the basis of the first-order solution, and then to determine their function $E(\mathbf{k})$ and recalculate the solution Eq. (48). The induced sources around the origin basically lead to a modification of the strength $\hat{M}(\mathbf{0})$, but not of the asymptotic decay. However, the far away points at the lines and planes could modify the asymptotic decay. A closer inspection of the field of these sources shows that it is of order $1/r^7$, since the differing rates lead to a local Laplacian acting on the first order-field decaying as $1/r^5$. Hence, every step of this perturbative calculation yields a leading term $1/r^5$; the amplitude changes in every step and its determination is a difficult problem indeed.

D. Correlations for general z

From the previous section, it is clear that correlations occur whenever the detailed balance condition is frustrated around the origin. The stationary master equation then becomes a discrete Poisson's equation in $(n-1)(d-1)$ dimensions, leading to correlations that decay with an integer power of the distance r. Following the derivation in Appendix B, it is clear that the asymptotic behavior comes from the lowest nonisotropic term in $E(\mathbf{k})$, since division by $1-D(\mathbf{k})\approx k^2$ gives a singularity. The value of the exponent can be calculated as

$$(n-1)(d-1)+O-2, \tag{50}$$

where $(n-1)(d-1)$ is the dimensionality of the correlation space and O is the order of the lowest nonisotropic terms in the expansion of $E(\mathbf{k})$. Although this result is remarkably simple, the actual calculation of $E(\mathbf{k})$ is not trivial, as it reflects the symmetries of the jump directions on the $(n-1)(d-1)$-dimensional lattice. Working out the four–dimensional lattice of the third-order moments in the fcc packing, we find that $O=2$ and correlations vanish as $1/r^4$.

VI. CORRELATED q DISTRIBUTIONS

So far, we have only discussed q distributions of the type Eq. (3), for which there are no correlations between q values at different sites. We have shown that, for these q distributions, there are no asymptotic two-point force correlations. In this section we will demonstrate that even the smallest correlation between q values at different sites induces two-point force correlations. We first solve the problem for arbitrary correlations in the triangular packing. Then, we study the fcc packing assuming only a nearest-neighbor q correlation; this already leads to force correlations that decay as $1/r^6$.

A. Triangular packing with arbitrary q correlations

In general, the (second-order) transition rates are defined by Eq. (31). For $z=2$, the displacement vector α can only take two values, for which we conveniently choose $\pm\frac{1}{2}$. This allows us to write the transition rates as

$$W_0(k)=\overline{q_{i,+1/2}q_{i+k,+1/2}}=\overline{q_{i,-1/2}q_{i+k,-1/2}},$$

$$W_{+1}(k)=\overline{q_{i,-1/2}q_{i+k,+1/2}}=\overline{q_{i,-1/2}(1-q_{i+k,-1/2})}$$
$$=1/2-W_0(k),$$

$$W_{-1}(k)=\overline{q_{i,+1/2}q_{i+k,-1/2}}=\overline{q_{i,+1/2}(1-q_{i+k,+1/2})}$$
$$=1/2-W_0(k). \tag{51}$$

Asymptotically, $W_0(k)$ has to approach the value 1/4, for q distributions without long-range correlations. As the second moments diffuse on a line, one can easily construct a detailed balance solution:

$$[1/2-W_0(k-1)]M(k-1)=[1/2-W_0(k)]M(k), \tag{52}$$

or

$$M(k)=\frac{1/2-W_0(0)}{1/2-W_0(k)}M(0). \tag{53}$$

This is the general form of the two-point force correlations $M(k)$ in the triangular packing, as a function of $W_0(k)$ that describes the q correlations. One can draw two interesting conclusions from this result. First of all, there can only be an uncorrelated solution if $W_0(k)$ is constant (i.e., 1/4) for each $k\neq 0$. This means that even the smallest q correlations lead to force correlations. Second, the long-distance behavior of the two-point force correlations is identical to that of the two-point q correlations, following from the simplicity of Eq. (53).

B. fcc packing with nearest-neighbor q correlations

Unfortunately, the analysis is much more complicated for the fcc packing, whose second-order moments live on the two-dimensional triangular lattice of Fig. 5. We, therefore, allow only correlations between q values at neighboring sites. Remember that one can easily construct an uncorrelated solution $M(\mathbf{r})$ for uncorrelated q distributions, Eq. (39), since all detailed balance conditions at the origin are identical by symmetry. This still holds when there are nearest-neighbor correlations. However, the detailed balance condition will now be frustrated on the ring of surrounding sites, as these are connected in four *a priori* different directions, see Fig. 5. In analogy to the problem discussed in the previous section, the stationary master equation for $\delta\hat{M}(\mathbf{r})$ transforms into

$$\delta\hat{M}(\mathbf{r})-1/6\sum_{\gamma\neq\mathbf{0}}\delta\hat{M}(\mathbf{r}-\boldsymbol{\gamma})=\rho(\mathbf{r}). \tag{54}$$

The "charge density" $\rho(\mathbf{r})$ is only nonzero around the frustrated ring, see Appendix C. Again, it is a discrete version of Poisson's equation, but now in two dimensions. The solution can, therefore, be expanded in cylindrical harmonics, $\exp(in\phi)/r^n$, and the sixfold symmetry of the lattice requires $n\geqslant 6$. The problem is again solved rigorously by Fourier transformation of Eq. (54). In Appendix C we show that

$$\delta\hat{M}(\mathbf{r})\propto\frac{\cos(6\phi)}{r^6}, \tag{55}$$

which is in agreement with the simple electrostatic picture.

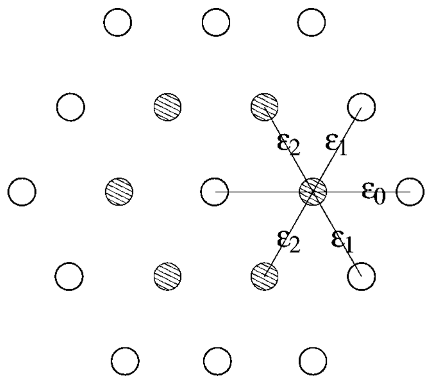

FIG. 5. fcc packing: second order moments diffuse on a triangular lattice. The ring around the origin has differing rates.

So, for the fcc packing, we find that even a nearest-neighbor q correlation leads to two-point force correlations that decay with a power law. This algebraic decay is generic for $z \geqslant 3$ since any q correlations lead to a master equation whose detailed balance relations cannot be solved around the origin.

VII. DISCUSSION

We have studied force correlations in the q model at infinite depth, for general q distributions. The calculated correlation functions are rather unusual: for q distributions of the type Eq. (3), correlations only show up at higher orders, and these correlations decay with a power of the distance. The only exceptions are the q distributions given by Eq. (17), which do produce a factorized force distribution. The results for the triangular packing and the fcc packing are summarized in Table I. As an example, consider two different sites i and $i+k$ in a layer of the triangular packing. Since there are no correlations in the second- and third-order force moments, we find $\langle f_i f_{i+k} \rangle = 1$ and $\langle f_i^2 f_{i+k} \rangle = \langle f^2 \rangle$, independent of the distance k. However, the moments $\langle f_i^3 f_{i+k} \rangle$ and $\langle f_i^2 f_{i+k}^2 \rangle$ are correlated and approach their asymptotic value as $1/k^5$. The fact that one has to go to higher orders to observe force correlations is the reason why numerical simulations only marginally differ from the mean field solutions [4]. The (single-site) mean field solutions $p^{\mathrm{mf}}(f)$ are correct up to the third-order moments, for the triangular packing. This implies that $p^{\mathrm{mf}}(f)$ ''wiggles'' around the real solution $p^*(f)$; the deviation $p^{\mathrm{mf}}(f) - p^*(f)$ changes its sign four times (Fig. 2).

TABLE I. Summary of the results for the triangular packing ($z=2$; $d=2$) and the fcc packing ($z=3$; $d=3$). The nth-order force moments diffuse on a $(n-1)(d-1)$-dimensional lattice; the lattice structures are listed in the first row. The second row shows the nature of the corresponding force correlations in the stationary state.

Packing	$n=2$	$n=3$	$n=4$	$n=2$, with q corr.
Triangular ($d=2$)	Line no corr.	Triangular no corr.	bcc $1/r^5$	Line like q corr.
fcc ($d=3$)	Triangular no corr.	4-dim. $1/r^4$	6-dim.	Triangular $1/r^6$

Packings that have more than three q values per site ($z \geqslant 3$) already have third-order correlations. Also this time correlations only decay algebraically; for the fcc packing we find $1/r^4$. This algebraic decay can be understood from an analogy with electrostatics. The force moments evolve according to a master equation, and the corresponding stationary state is described by a discrete version of Poisson's equation. The ''source'' turns out to be a multipole around the origin, which is only active whenever the master equation has no simple detailed balance solution. The moments therefore approach their asymptotic (uncorrelated) values algebraically. The value of the exponent depends on the dimension of the correlation space $(n-1)(d-1)$, and on the symmetry of the multipole, see Eq. (50).

Although in general correlations do exist, there is a special class of q distributions, given by Eq. (17), for which there are no force correlations at all. This has been demonstrated by means of condition (14), which has a nice physical interpretation. It can be shown that the function $\psi(s)$ is the Laplace transform of the distribution of *interparticle forces* that live on the bonds connecting the particles: $v_{i,\alpha} = q_{i,\alpha} f_i$. Although the q's leaving a site are correlated (they have to add up to 1), the corresponding $v_{i,\alpha}$ can become statistically independent. It is only when this miracle happens that the force distribution becomes a product state. Nevertheless, the q distributions for which this is the case range from infinitely sharp ($r \to \infty$) to almost critical ($r \to 0$). It is interesting to note that a similar calculation has been done recently for the asymmetric random average process (ARAP) [16]. This $1+1$ dimensional model maps onto the q-model with triangular packing, with a broken left–right symmetry. The extension of our calculation to asymmetric $\eta(\vec{q})$ is straightforward [10]: one has to replace r by r_α, and zr by $\Sigma_\alpha r_\alpha$ in Eqs. (16)–(19).

Finally, we found that there will be two-point force correlations whenever the q values of different sites are correlated. Even with only nearest-neighbor q correlations, the fcc packing has force correlations that vanish as $1/r^6$. Again, the triangular packing is less sensitive for correlations; the nature of the force correlations is identical to that of the q correlations, Eq. (53).

ACKNOWLEDGMENTS

The authors would like to thank Wim van Saarloos, Martin van Hecke, and Martin Howard for stimulating discussions.

APPENDIX A: MOMENTS OF q DISTRIBUTIONS

This appendix is about the moments of the q distributions, defined by

$$\overline{q_1^{n_1}q_2^{n_2}\cdots q_z^{n_z}}=\int \eta(\vec{q})\,\delta\!\left(1-\sum_{\alpha} q_\alpha\right)d\vec{q}\,q_1^{n_1}q_2^{n_2}\cdots q_z^{n_z}. \tag{A1}$$

These different moments are not independent because of the δ constraint. As the distributions are normalized, the zeroth-order moments are unity; the first-order moments are, of course, all $1/z$. All second-xorder moments, for which $\Sigma_i n_i=2$, can be described by only one free parameter. Defining η_n as

$$\eta_n=\int \eta(\vec{q})\,\delta\!\left(1-\sum_{\alpha} q_\alpha\right)d\vec{q}\,q_1^{n}, \tag{A2}$$

one finds

$$\sum_{i=1}^{z}\overline{q_1q_i}=\eta_2+(z-1)\overline{q_1q_2}$$
$$=\int \eta(\vec{q})\,\delta\!\left(1-\sum_{\alpha} q_\alpha\right)d\vec{q}\,q_1\sum_{i=1}^{z} q_i=1/z, \tag{A3}$$

hence

$$\overline{q_1q_2}=\frac{1}{(z-1)}(1/z-\eta_2). \tag{A4}$$

From a similar argument, one can derive for the third-order moments

$$\overline{q_1^3}=\eta_3,\quad \overline{q_1^2q_2}=\frac{1}{(z-1)}(\eta_2-\eta_3). \tag{A5}$$

For $z=2$ there is even a relation between η_3 and η_2:

$$1=\sum_{ijk}\overline{q_iq_jq_k}=2\,\eta_3+6\overline{q_1^2q_2}\Rightarrow \eta_3=\frac{3}{2}\,\eta_2-1/4. \tag{A6}$$

For $z=3$, there is an additional third moment, namely,

$$1=\sum_{ijk}\overline{q_iq_jq_k}=3\,\eta_3+18\overline{q_1^2q_2}+6\overline{q_1q_2q_3}$$
$$\Rightarrow\overline{q_1q_2q_3}=\frac{1}{6}(1-9\,\eta_2+6\,\eta_3). \tag{A7}$$

The extension to higher orders and higher z is straightforward.

For the special class of $\eta(\vec{q})$ defined in Eq. (17), one can calculate the moments η_n from a generalization of Eq. (16) [10],

$$\eta_n=\frac{\Gamma(zr)\Gamma(r+n)}{\Gamma(r)\Gamma(zr+n)}. \tag{A8}$$

In order to show that Eq. (24) is indeed obeyed by the special class (with $z=2$), we first invert Eq. (A8) for $n=2$,

TABLE II. The transition rates $W_\gamma(\mathbf{r})$ for the fourth-order master equation.

	From \ to	(0,0,0)	$(k,0,0)$	$(k,k,0)$	$(k,l,0)$	(k,l,m)
Origin	(0,0,0)	$\overline{q_1^4}$	$\overline{q_1^3q_2}$	$\overline{q_1^2q_2^2}$		
Line (c)	$(k,0,0)$ (k,k,k)	$\frac{1}{2}\overline{q_1^3}$	$\frac{1}{2}\overline{q_1^3}$	$\frac{1}{2}\overline{q_1^2q_2}$	$\frac{1}{2}\overline{q_1^2q_2}$	
Line (b)	$(k,k,0)$	$(\overline{q_1^2})^2$	$\overline{q_1^2}\,\overline{q_1q_2}$	$(\overline{q_1^2})^2$	$\overline{q_1^2}\,\overline{q_1q_2}$	$(\overline{q_1q_2})^2$
Plane	$(k,l,0)$ (k,k,l)		$\frac{1}{4}\overline{q_1^2}$	$\frac{1}{4}\overline{q_1^2}$	$\frac{1}{4}\overline{q_1^2}$	$\frac{1}{4}\overline{q_1q_2}$
Bulk	(k,l,m)			$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

$$r=\frac{1-2\,\eta_2}{4\,\eta_2-1}. \tag{A9}$$

From this one can calculate η_4 as a function of η_2, which precisely results in Eq. (24). A similar inversion for $z=3$ leads to

$$r=\frac{1-3\,\eta_2}{9\,\eta_2-1}, \tag{A10}$$

from which one derives Eq. (25).

APPENDIX B: THE bcc LATTICE

In the triangular packing, the fourth-order force moments $\langle f_i f_{i+k} f_{i+l} f_{i+m}\rangle$ diffuse on the bcc lattice of Fig. 4, with differing jump rates on special lines and planes. In this appendix, we list these rates explicitly and we solve the corresponding stationary master equation.

The jump rates can be calculated from

$$W_\gamma(k,l,m)=\overline{q_{i,\alpha}\,q_{i+k,\alpha'}\,q_{i+l,\alpha''}\,q_{i+k,\alpha'''}}, \tag{B1}$$

with the $z^4=16$ jump directions

$$\boldsymbol{\gamma}=(\alpha'-\alpha,\alpha''-\alpha,\alpha'''-\alpha). \tag{B2}$$

As α can take the values $\pm\frac{1}{2}$, there are two *self-rates* for which all α's are the same. As a consequence, there are 14 outgoing directions, namely, $\pm(1,0,0)$, $\pm(1,1,1)$, and $\pm(1,1,0)$ plus their permutations. The first two are directions for which three of the four α's are equal, and they correspond to the *corners* of Fig. 4; the third represents the jumps towards the *body centers*. If all position indices in Eq. (B1) are different, the transition rates are simply $1/z^4=1/16$. On the special lines and planes where one or more position indices coincide, we encounter differing rates. The geometry of the problem is summarized in Table II.

From this table we deduce the rates ε_c to the corners and ε_b to the body centers, which occur in relation (46). We find

$$\varepsilon_c=\frac{\overline{q_1^3 q_2}}{1-2\overline{q_1^4}}-\frac{1}{14},\quad \varepsilon_b=\frac{\overline{q_1^2 q_2^2}}{1-2\overline{q_1^4}}-\frac{1}{14}, \tag{B3}$$

and one easily verifies from the property $q_1+q_2=1$ that the relation $8\varepsilon_c+6\varepsilon_b=0$ holds. In general, the rates do not obey the detailed balance condition Eq. (38) in the elementary loop *origin–corner–body-center–origin*. Keeping only the rates in this loop as deviations from the bulk leads to Eq. (47). For the definition of the two functions $E(\mathbf{k})$ and $D(\mathbf{k})$ we introduce two auxiliary functions: one for the contribution of the corners

$$\tilde{E}_c(\mathbf{k})=\frac{1}{4}\left[\cos\frac{k_x+k_y+k_z}{2}+\cos\frac{k_x-k_y+k_z}{2}+\cos\frac{k_x+k_y-k_z}{2}+\cos\frac{k_x-k_y-k_z}{2}\right] \tag{B4}$$

and one related to the body centers

$$\tilde{E}_b(\mathbf{k})=\frac{1}{3}(\cos k_x+\cos k_y+\cos k_z). \tag{B5}$$

The two functions $\tilde{D}(\mathbf{k})$ and $\tilde{E}(\mathbf{k})$ are then given as

$$\tilde{D}(\mathbf{k})=\frac{4}{7}\tilde{E}_c(\mathbf{k})+\frac{3}{7}\tilde{E}_b(\mathbf{k}),$$

$$\tilde{E}(\mathbf{k})=\varepsilon[\tilde{E}_c(\mathbf{k})-\tilde{E}_b(\mathbf{k})], \tag{B6}$$

with $\varepsilon=8\varepsilon_c=-6\varepsilon_b$.

For the large $\mathbf{r}$ behavior we need the expansions for small k. One finds

$$\tilde{E}_c(\mathbf{k})=1-\frac{1}{8}k^2+\frac{1}{384}[k^4+4(k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2)]+\cdots \tag{B7}$$

and

$$\tilde{E}_b(\mathbf{k})=1-\frac{1}{6}k^2+\frac{1}{72}[k^4-(k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2)]+\cdots. \tag{B8}$$

From these expressions one derives the expansion

$$\frac{\tilde{E}(\mathbf{k})}{1-\tilde{D}(\mathbf{k})}=\frac{7\epsilon}{24}\left(1-\frac{7}{32}k^2+\frac{7}{8}\,\frac{k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2}{k^2}+\cdots\right). \tag{B9}$$

The first two terms in the expansion are regular and thus give rise to short-range contributions. The last term leads to the asymptotic behavior, by means of the inverse Fourier transform

$$\int\frac{d\mathbf{k}}{V_{\mathrm{BZ}}}\,\frac{(k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2)\exp(-i\mathbf{k}\cdot\mathbf{r})}{k^2}. \tag{B10}$$

This integral can be evaluated by differentiation of the well-known equation

$$\int\frac{d\mathbf{k}}{V_{\mathrm{BZ}}}\,\frac{\exp(-i\mathbf{k}\cdot\mathbf{r})}{k^2}\simeq\frac{1}{4\pi r}, \tag{B11}$$

where a factor k_x in Eq. (B10) corresponds to applying $\partial/\partial x$. This leads to expression (49).

APPENDIX C: *q* CORRELATIONS IN THE fcc PACKING

In Eq. (54) we formulated the problem for the second moments in the fcc packing with nearest-neighbor q correlations. The "charge density" $\rho(\mathbf{r})$ on the right-hand side of the equation is the product of the moment $\hat{M}(\boldsymbol{\gamma})$, referring to the neighbors of the origin (all are the same by symmetry), with a function whose Fourier transform is given by

$$\tilde{E}(\mathbf{k})=\sum_{\gamma',\gamma}w_{\gamma-\gamma'}\exp[i\mathbf{k}\cdot(\boldsymbol{\gamma}+\boldsymbol{\gamma}')]. \tag{C1}$$

The $w_{\gamma-\gamma'}$ are the deviations from the bulk transition rates 1/6. These are only nonzero for the ring of nearest neighbors around the origin shown in Fig. 5,

$$w_0=-\varepsilon_0,\quad w_1=w_5=-\varepsilon_1,$$

$$w_2=w_4=-\varepsilon_2,\quad w_3=\varepsilon_0+2\varepsilon_1+2\varepsilon_2. \tag{C2}$$

The equalities reflect the symmetry of the triangular lattice. Inserting Eq. (C1) into the Fourier transform of Eq. (54) leads to

$$\hat{M}(\mathbf{r})=2/3+\hat{M}(\boldsymbol{\gamma})\sum_{\mathbf{k}}\frac{\tilde{E}(\mathbf{k})}{1-\tilde{D}(\mathbf{k})}\exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{C3}$$

The consistency equation for $\hat{M}(\boldsymbol{\gamma})$ follows by taking $\mathbf{r}$ as one of the nearest neighbors of the origin. The function $\tilde{D}(\mathbf{k})$ is given by

$$\tilde{D}(\mathbf{k})=\frac{1}{3}\left(\cos k_x+\cos\frac{k_x+\sqrt{3}k_y}{2}+\cos\frac{k_x-\sqrt{3}k_y}{2}\right), \tag{C4}$$

and $\tilde{E}(\mathbf{k})$ can be expressed as

$$\tilde{E}(\mathbf{k})/6=\varepsilon_0[1-\tilde{D}(2\mathbf{k})]+2\varepsilon_1[1-\tilde{D}'(\mathbf{k})]+2\varepsilon_2[1-\tilde{D}(\mathbf{k})], \tag{C5}$$

with the new function

$$\tilde{D}'(\mathbf{k})=\tilde{D}(\sqrt{3}\,k_y,\sqrt{3}\,k_x). \tag{C6}$$

For the asymptotic behavior of $\hat{M}(\mathbf{r})$ we must make an expansion of $\tilde{E}(\mathbf{k})/[1-\tilde{D}(\mathbf{k})]$. For the first two terms we find

$$x\frac{1-\tilde{D}(2\mathbf{k})}{1-\tilde{D}(\mathbf{k})}=4\left(1-\frac{3}{16}k^2+\frac{3}{256}k^4+\frac{1}{192}\frac{k_x^6-6k_x^4k_y^2+9k_x^2k_y^4}{k^2}+\cdots\right), \quad \text{(C7)}$$

$$\frac{1-\tilde{D}'(\mathbf{k})}{1-\tilde{D}(\mathbf{k})}=3\left(1-\frac{1}{8}k^2+\frac{1}{128}k^4+\frac{1}{288}\frac{k_x^6-6k_x^4k_y^2+9k_x^2k_y^4}{k^2}+\cdots\right), \quad \text{(C8)}$$

and the third term is simply a constant. The asymptotic behavior is given by Fourier inversion of the first singular term in $\mathbf{k}$, i.e.,

$$\int\frac{d\mathbf{k}}{V_{BZ}}\frac{(k_x^6-6k_x^4k_y^2+9k_x^2k_y^4)\exp(-i\mathbf{k}\cdot\mathbf{r})}{k^2}\simeq\frac{960}{\pi}\frac{\cos(6\phi)}{r^6}. \quad \text{(C9)}$$

This integral can be obtained by differentiation of

$$\int\frac{d\mathbf{k}}{V_{BZ}}\frac{\exp(-i\mathbf{k}\cdot\mathbf{r})}{k^2}\simeq\frac{\ln(L/r)}{2\pi}, \quad \text{(C10)}$$

where L is the size of the system.

[1] C. Liu *et al.*, Science **269**, 513 (1995).

[2] D.M. Mueth, H.M. Jaeger, and S.R. Nagel, Phys. Rev. E **57**, 3164 (1998); D.L. Blair *et al.*, *ibid.* **63**, 041304 (2001).

[3] G. Løvoll, K.J. Måløy, and E.G. Flekkøy, Phys. Rev. E **60**, 5872 (1999).

[4] S.N. Coppersmith *et al.*, Phys. Rev. E **53**, 4673 (1996).

[5] P. Claudin and J-P. Bouchaud, Phys. Rev. Lett. **78**, 231 (1997); M. Nicodemi, *ibid.* **80**, 1340 (1998); J.E.S. Socolar, Phys. Rev. E **57**, 3204 (1998); M.L. Nguyen and S. N. Coppersmith, *ibid.* **59**, 5870 (1999); O. Narayan, *ibid.* **63**, 010 301 (2000).

[6] In fact, the probability for large forces decays even faster than exponentially $\sim\exp(-af^{\beta})$ whenever there is a maximum q value <1. The value of β can be related to $q_{\max}$ (in a mean field approximation) as $\beta=\ln(z)/\ln(zq_{\max})$, where z is the number of q values per site. J-P. Bouchaud *et al.*, Phys. Rev. E **57**, 4441 (1998).

[7] R. Rajesh and S.N. Majumdar, Phys. Rev. E **62**, 3186 (2000).

[8] M. Lewandowska, H. Mathur, and Y.-K. Yu, Phys. Rev. E **64**, 026107 (2001).

[9] J.H. Snoeijer and J.M.J. van Leeuwen, e-print cond-mat/0110230.

[10] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1990), p. 214 [Eq. (9.39)].

[11] The limit $r\to0$ is a route towards the critical distribution because the q values become increasingly concentrated around 0 and 1. Although the distribution of Eq. (17) is not normalizable for $r=0$, all higher-moments approach the moments of the critical distribution for $r\to0$, see Appendix A. The corresponding force distributions approach $1/f$ with a cutoff for large forces.

[12] The remaining equation for ψ_2 cannot be solved if $\eta_2=1/z$, i.e., for the critical q distribution. As the corresponding force distribution has diverging moments, one cannot even Taylor expand $\tilde{p}^*(s)$ and $\tilde{\psi}(s)$.

[13] N.W. Ashcroft and N.D. Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976), p. 73 (Fig. 4.13).

[14] For an uncorrelated asymptotic state all $M(\mathbf{r})=1$, provided that no indices coincide nor are equal to zero. Any deviation from 1, i.e., $\delta M(\mathbf{r})\neq0$, indicates correlations. The same holds for the "hatted" variables, up to a factor $1-2/16=7/8$ that comes from the transformation.

[15] A *discrete* Poisson's equation with a multipole source term can indeed have a short-range solution. This is the case if $E(\mathbf{k})=K(\mathbf{k})[1-D(\mathbf{k})]$, and $K(\mathbf{k})$ is a regular function.

[16] F. Zielen and A. Schadschneider, J. Stat. Phys. **106**, 173 (2002).